J. Fluid Mech. (1978), vol. 88, part 1, pp. 181–203
Printed in Great Britain

Wave propagation in a viscoelastic tube containing a viscous fluid

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(Received 16 July 1977 and in revised form 6 March 1978)

Small amplitude, axially symmetric waves in a thin-walled viscoelastic tube containing a viscous compressible fluid are considered. Previous authors have found two modes of propagation for such waves but have studied them only in the low frequency, long wavelength limit. We show that there are infinitely many modes and study them at all frequencies. The appropriate dispersion equation was derived previously (Rubinow & Keller 1971) and analysed for an inviscid fluid. Now it is analysed for a viscous fluid. Asymptotic formulae for the propagation constant k are obtained for both low and high frequencies and for various ranges of the parameters characterizing the tube and the fluid. Special attention is paid to the case of a rigid tube and to parameter values that characterize the flow of blood in mammalian arteries. In addition, numerical results are obtained which complement the asymptotic formulae. Graphs of the velocity c *vs.* the frequency ω are presented for various modes and for various ranges of the parameters. Transmission-line equations and formulae for the impedance and compliance of the fluid–tube system are obtained, together with asymptotic and numerical results.

1. Introduction

There are infinitely many modes of propagation for waves in fluid-filled elastic tubes, such as water pipes and blood vessels. However only the first two modes have been found by previous investigators, and then only in the low frequency, long wavelength limit. We have found the higher modes as well and studied all of them over the entire range of frequencies. In doing so we have taken into account the viscosity and compressibility of the fluid, the elasticity and viscoelasticity of the tube wall, and the external constraints on the wall. We have used both analytical and numerical means in our previous work (Rubinow & Keller 1971, to be referred to as I). There we presented results for axially symmetric modes in a thin-walled tube of circular cross-section containing an *inviscid* fluid. Here we do the same for a *viscous* fluid.

The theoretical analysis of waves in fluid-filled elastic tubes was begun by Young (1808) and continued by Weber (1866), Résal (1876), Korteweg (1878), Lamb (1898), Womersley (1957) and many others. They were concerned with blood flow and the blood pressure pulse in mammalian arteries. For this application the low frequency, long wavelength approximation is usually valid and only the two lowest modes propagate, the others being evanescent. However even in this case the higher modes play a role at junctions, bifurcations, bends, valves, etc., as is well known from the theory of waveguides. At high frequencies the higher modes can propagate in any tube. Therefore they will occur in other cases of wave propagation in fluid-filled tubes.

Since we shall examine all the modes at all frequencies, we shall incidentally recover the known results for the first two modes at low frequencies. But even for this case we shall obtain some new results, such as the low frequency behaviour of the second mode and the explicit dependence of the first two propagation speeds on the viscoelastic parameter of the material. We shall also clarify the confusion in the literature which results from the hitherto unrecognized non-uniform dependence of the propagation velocities upon the viscosity coefficient μ and the angular frequency ω at $\mu = 0$, $\omega = 0$. In addition we shall show that Womersley's parameter $\alpha\omega$ is inadequate to characterize waves except at *low* frequencies in *large* arteries. His approximation is not accurate for arterioles and capillaries even at low frequencies. We find instead that at both very low and very high frequencies the results depend upon the viscosity as well as upon $\alpha\omega$.

For the limiting case of a rigid tube we shall obtain high frequency asymptotic formulae for all the modes. The first two of these modes were found by Rayleigh and Helmholtz at low frequencies. For any tube, we shall present transmission-line equations with apparently new formulae for the impedance, compliance, etc. valid at all frequencies, together with asymptotic expansions of them. These equations are useful in the study of blood flow (Rubinow & Keller 1968), where the average properties over cross-sections are measured rather than the detailed velocity distribution.

All of our results are derived for a tube which, in the absence of waves, is a circular cylindrical shell of constant thickness, made of a viscoelastic material and filled with a compressible viscous fluid at rest. The outer surface of the tube is assumed to be partially constrained. In I we obtained axially symmetric wavelike solutions of the linearized equations and boundary conditions governing the motion of the fluid and the tube wall. These were the Navier–Stokes equations and the equations of viscoelasticity respectively, together with continuity conditions at the fluid–solid interface and a constraint involving a complex impedance matrix at the outer surface. The solutions were proportional to $e^{i(kz-\omega t)}$, where z is distance along the tube axis, t is time, k is the propagation constant and ω is the angular frequency. The dispersion equation, which relates k and ω, was derived for the practically important case of a thin-walled tube.

Corresponding to each root $k(\omega)$ of this equation there is a solution which is called a mode of propagation. In I we presented asymptotic formulae for these roots at both low and high frequencies and numerical solutions at intermediate frequencies for an inviscid fluid. We also gave the phase velocity, the group velocity and the ratio of the amplitudes of the radial and axial components of the tube wall velocity for each mode. Now we shall present similar results for the modes in a viscous fluid and also give their impedances and compliances.

In the next section (§2), we describe the formulation of the problem as given in I, introduce the notation and present the dispersion equation. In §3 we analyse this equation for the special case of a rigid tube. In §4 we analyse it for an unconstrained elastic tube. In §5 we show how the results of §4 can be applied to an unconstrained viscoelastic tube. In §6 we derive transmission-line equations and expressions for the impedance and compliance, and analyse these expressions. Graphs of various quantities based upon our results are presented throughout the paper.

2. The dispersion equation

Let us consider a circular cylindrical tube of wall thickness h and inner radius a composed of a viscoelastic material of density ρ_1, complex Young's modulus E and complex Poisson ratio σ. The viscous behaviour of this material is accounted for by the imaginary parts of E and σ. When E and σ are real, the tube material is elastic. Let the tube be filled with a viscous fluid of density ρ_0, shear and bulk viscosity coefficients μ and μ', and sound speed c_0. We shall treat axially symmetric motions of the tube wall and fluid. Therefore we let ξ' and ζ' respectively denote the longitudinal and radial components of the displacement of the tube material, let w' and u' respectively denote the longitudinal and radial components of velocity of the fluid, and let p' denote the excess pressure in the fluid.

Now ξ' and ζ' satisfy the equations of viscoelasticity, while u', w' and p' satisfy the Navier–Stokes equations. At the inner boundary of the tube wall, the velocity and normal stress must be continuous. The outer boundary is assumed to be partially constrained. This is represented by requiring the displacement vector to be related to the velocity vector by a complex impedance matrix with elements Z'_{ij} $(i,j = 1,2)$ which characterizes the material outside the tube.

A solution of these equations is the state of rest with the tube material undisplaced, so that $u' = v' = p' = \xi' = \zeta' = 0$. To describe wave propagation we linearize the problem around this state of rest and also simplify the equations of motion of the tube wall for the case of a thin wall, i.e. $h \ll a$. Then we introduce the radial co-ordinate r', axial co-ordinate z', time t', angular frequency ω' and propagation constant k'. We also define the following dimensionless quantities:

$$\left.\begin{aligned} r = r'/a, \quad z = z'/a, \quad t = \omega_0 t', \quad \xi = \xi'/a, \quad \zeta = \zeta'/a, \quad c_0 = c_0'/a\omega_0, \quad k = k'a,\\ \omega = \omega'/\omega_0, \quad u = u'/a\omega_0, \quad w = w'/a\omega_0,\\ p = p'/\rho_0 a^2\omega_0^2, \quad Z_{ij} = Z'_{ij}/\omega_0 k\rho_1. \end{aligned}\right\} \quad (2.1)$$

The frequency ω_0 in (2.1) is defined as follows, together with the parameters κ, ϕ, α, α', m, Ω and γ, which will be needed later:

$$\left.\begin{aligned} \omega_0 = |E/\rho_1 a^2(1-\sigma^2)|^{\frac{1}{2}}, \quad \kappa^2 = k^2 - \omega^2(c_0^2 - i\omega\phi/\alpha)^{-1}, \quad \phi = 1 + \alpha/\alpha',\\ \alpha = \rho_0 a^2\omega_0/\mu, \quad \alpha' = \rho_0 a^2\omega_0/\mu', \quad m = \rho_1 h/\rho_0 a,\\ \Omega = E/\rho_1 a^2(1-\sigma^2)\,\omega_0^2 \equiv e^{-i\gamma}. \end{aligned}\right\} \quad (2.2)$$

In I we showed that, as a consequence of causality, the viscoelastic parameter γ lies in the range $0 \leqslant \gamma \leqslant \pi$.

We seek a solution of the linearized equations of motion of the fluid and of the thin-walled tube of the form

$$[p, u, w, \xi, \zeta] = [p_1(r), u_1(r), w_1(r), \xi_0, \zeta_0]\, e^{i(kz-\omega t)}. \quad (2.3)$$

Here ξ_0 and ζ_0 are constants. We find that p_1, u_1 and w_1 are given by

$$\left.\begin{aligned} p_1(r) &= p_0 I_0(\kappa r),\\ u_1(r) &= w_0 \frac{ik}{(k^2 - i\alpha\omega)^{\frac{1}{2}}} I_1[r(k^2 - i\alpha\omega)^{\frac{1}{2}}] + p_0 \frac{i\omega\kappa}{c_0(\kappa^2 - k^2)} I_1(\kappa r),\\ w_1(r) &= w_0 I_0[r(k^2 - i\alpha\omega)^{\frac{1}{2}}] - p_0 \frac{\omega k}{c_0^2(\kappa^2 - k^2)} I_0(\kappa r). \end{aligned}\right\} \quad (2.4)$$

The boundary conditions at the inner and outer surfaces of the tube wall lead to a set of four homogeneous linear algebraic equations for the four constants p_0, w_0, ξ_0 and ζ_0. In order that these equations have a non-trivial solution, the determinant of the coefficient matrix must vanish. This yields a transcendental equation, called the dispersion equation, which can be viewed as determining k in terms of ω. It is given by equation (56) of I, which has minor errors which resulted from the replacement of the factor ωk in (55) of I by $\omega\kappa$. The correct dispersion equation is

$$\begin{aligned}
&I_0[(k^2-i\alpha\omega)^{\frac{1}{2}}]\Big(-I_0(\kappa)\,\omega^2 m(k^2\Omega-\omega^2+Z_{22})\\
&+\kappa I_1(\kappa)\left\{\left[m(\Omega-\omega^2+Z_{11})+i\frac{2\omega}{\alpha}\right]m(k^2\Omega-\omega^2+Z_{22})\right.\\
&\left.-\left[m(k\sigma\Omega-iZ_{12})+i\frac{2\omega}{\alpha}k\right]\left[m(k\sigma\Omega+iZ_{21})+i\frac{2\omega}{\alpha}k\right]\right\}\Big)\\
&-\frac{I_1[(k^2-i\alpha\omega)^{\frac{1}{2}}]}{(k^2-i\alpha\omega)^{\frac{1}{2}}}\left[-I_0(\kappa)\,\omega^2\left\{\omega^2+k\left[m(k\sigma\Omega+iZ_{21})+i\frac{2\omega}{\alpha}k\right]\right\}\right.\\
&+kI_0(\kappa)\left(k\left[m(\Omega-\omega^2+Z_{11})+i\frac{2\omega}{\alpha}\right]m(k^2\Omega-\omega^2+Z_{22})\right.\\
&\left.-\left[m(k\sigma\Omega-iZ_{12})+i\frac{2\omega}{\alpha}k\right]\left\{\omega^2+k\left[m(k\sigma\Omega+iZ_{21})+i\frac{2\omega}{\alpha}k\right]\right\}\right)\\
&\left.+\omega^2\kappa I_1(\kappa)\left(m(\Omega-\omega^2+Z_{11})+i\frac{2\omega}{\alpha}\right)\right] = 0. \qquad (2.5)
\end{aligned}$$

For each ω, there are an infinite number of solutions k of (2.5), which occur in pairs $\pm k$ because the equation is even in k. Corresponding to each solution k, there is a solution $[p_0, w_0, \xi_0, \zeta_0]$ of the linear equations, which is determined up to a single constant factor. When this solution is used in (2.3) and (2.4) it yields a particular mode of oscillation. Those modes for which $|\mathrm{Im}\,k|$ is large decay rapidly with distance along the tube, so they are called non-propagating. The modes with $\mathrm{Im}\,k = 0$ or $|\mathrm{Im}\,k|$ small are the propagating modes, which can carry energy over long distances along the tube. In I we showed that for an inviscid fluid the propagating modes consist of two tube modes, which are the only ones that propagate at low frequencies, plus an infinite number of acoustic modes, which propagate only at high frequencies.

3. Viscous incompressible fluid in a rigid tube

The quantity m, defined in (2.2), is half the ratio of the tube mass to the fluid mass per unit length of tube. When $m \to \infty$, (2.5) reduces to the dispersion equation for a viscous compressible fluid in a rigid tube, which is

$$\{k^2-\omega^2(c_0^2-i\omega\phi/\alpha)^{-1}\}F[\{k^2-\omega^2(c_0^2-i\omega\phi/\alpha)^{-1}\}^{\frac{1}{2}}] = k^2F[(k^2-i\alpha\omega)^{\frac{1}{2}}]. \qquad (3.1)$$

Here F is defined in terms of Bessel functions by

$$F(x) = 2I_1(x)/xI_0(x). \qquad (3.2)$$

We shall now obtain asymptotic formulae for the roots k of (3.1) in terms of the parameters in (3.1).

Let us consider first the low frequency case in which $\omega \ll 1$ with the other parameters fixed. To find solutions for which $|k| \ll 1$, we use the following expansion of F:

$$F(z) = 1 - \tfrac{1}{8}z^2 + \tfrac{1}{48}z^4 + O(z^6), \quad |z| \ll 1. \tag{3.3}$$

Upon using (3.3) in (3.1), retaining the terms of lowest degree in ω and k, then solving for k, we obtain

$$k = (8\omega/\alpha c_0^2)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} + \ldots, \quad \omega \ll 1. \tag{3.4}$$

Considering the other parameters shows that (3.4) holds when $\alpha\omega \ll 1$, $\omega/c_0 \ll 1$ and $\omega/\alpha c_0^2 \ll 1$. The result (3.4) is due to Rayleigh (1945, p. 327).

Next, we suppose that $\omega \ll 1$ but that $|k|$ is not small. We expand (3.1) for ω small and retain terms of lowest degree in ω, which yields

$$kF'(k) = 0, \quad \omega \ll 1. \tag{3.5}$$

The root $k = 0$ is spurious, since it violates the hypotheses under which (3.5) was derived. To solve (3.5) for $|k|$ large, we use the following asymptotic expansion of F:

$$F(z) = \frac{2}{iz}\tan(iz + \tfrac{1}{4}\pi)\left\{1 + \frac{1}{8iz}[3\cot(iz + \tfrac{1}{4}\pi) - \tan(iz + \tfrac{1}{4}\pi)] + O(z^{-2})\right\},$$

$$|z| \gg 1, \quad -\tfrac{1}{2}\pi < \arg z < \tfrac{3}{2}\pi. \tag{3.6}$$

When $|\mathrm{Re}\, z| \to \infty$, (3.6) becomes

$$F(z) = \frac{2}{z}(\mathrm{sgn}\,\mathrm{Re}\, z) + \frac{1}{z^2} + O(z^{-3}), \quad |\mathrm{Re}\, z| \gg 1. \tag{3.7}$$

Using (3.6) in (3.5) gives $\sin(2ik + \frac{1}{2}\pi) = 2ik + \ldots$. If $\mathrm{Re}\, k \gg 1$, this equation simplifies further to $\exp(2k - \frac{1}{2}i\pi) = 4k + \ldots$. The solutions of this equation are given asymptotically by

$$k \sim \tfrac{1}{2}\log[(2n+1)2\pi] + \tfrac{1}{2}i(2n+1)\pi, \quad n = 1, 2, \ldots, \quad \omega \ll 1. \tag{3.8}$$

Re-examination of the above derivation shows that (2.8) is valid if $\alpha^2 c_0^2/\phi \ll \alpha\omega \ll |k|^2$. Except for the factor of 2 in the denominator of the imaginary part, this result was obtained by Fitz-Gerald (1972) and is in excellent agreement with his numerical calculation of the first ten roots of (3.5). He has pointed out that the complex conjugate of each root of (3.5) is also a root, but that these conjugates are not asymptotic to roots of (3.1). However, the negative of each root of (3.1) is also a root, as we explained in I.

Now we turn to the case $\alpha\omega \gg 1$. There are two simplifying limits of (3.1) to consider: when the left side is large and the right side is a perturbation and vice versa. If we neglect the right side of (3.1), we obtain the results

$$k = \left(\frac{\omega^2}{c_0^2 - i\omega\phi/\alpha}\right)^{\frac{1}{2}}, \quad k = \left(\frac{\omega^2}{c_0^2 - i\omega\phi/\alpha} - \beta_{1n}^2\right)^{\frac{1}{2}} \quad \text{for} \quad n = 1, 2, \ldots.$$

Here β_{1n} is the nth positive zero of the Bessel function J_1, i.e. $J_1(\beta_{1n}) = 0$. When, in addition, $\omega \ll \alpha c_0^2/\phi$, these become $k = \omega/c_0 + \ldots$ and $k = (\omega^2/c_0^2 - \beta_{1n}^2)^{\frac{1}{2}} + \ldots$, which are just the results for an inviscid compressible fluid in a rigid tube.

To take account of the right side of (3.1), we invoke (3.7) so that $F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]$ is replaced by $2[\omega^2/(c_0^2 - i\omega\phi/\alpha) - \beta_{1n}^2 - i\alpha\omega]^{-\frac{1}{2}}$. Then we obtain

$$k = \frac{\alpha\omega}{(\alpha^2 c_0^2 - i\alpha\omega\phi)^{\frac{1}{2}}}\left\{1 + \frac{e^{\frac{1}{4}i\pi}}{(\alpha\omega)^{\frac{1}{2}}}\left[\frac{\alpha^2 c_0^2 - i\alpha\omega\phi}{\alpha^2 c_0^2 - i\alpha\omega(\phi - 1)}\right]^{\frac{1}{2}} + \ldots\right\}, \tag{3.9}$$

$$k = \left(\frac{\alpha^2\omega^2}{\alpha^2c_0^2 - i\alpha\omega\phi} - \beta_{1n}^2\right)^{\frac{1}{2}} \left\{1 + \left[-i\alpha\omega\frac{\alpha^2c_0^2 - i\alpha\omega(\phi-1)}{\alpha^2c_0^2 - i\alpha\omega\phi} - \beta_{1n}^2\right]^{-\frac{1}{2}} + \ldots\right\}. \quad (3.10)$$

In deriving (3.10), we used the identity

$$F'(z) = 2[1 - F(z)]/z - \tfrac{1}{2}zF^2(z). \quad (3.11)$$

If the frequency ω is not too large, so that $\omega \ll \alpha c_0^2/\phi$, then (3.9) and (3.10) simplify to

$$k = \omega c_0^{-1}[1 + (\alpha\omega)^{-\frac{1}{2}} e^{\frac{1}{4}i\pi} + \ldots], \quad (3.12)$$

$$k = (\omega^2 c_0^{-2} - \beta_{1n}^2)^{\frac{1}{2}} [1 + (-i\alpha\omega - \beta_{1n}^2)^{-\frac{1}{2}} + \ldots], \quad n = 1, 2, 3, \ldots. \quad (3.13)$$

The result (3.12) was given by Kirchhoff (1868; see also Rayleigh 1945, p. 325). The leading term in (3.13) is the inviscid result of I and Redwood (1961), while the correction term appears to be new. From (3.12) it follows that the phase velocity $c = \omega/\text{Re}\,k$ is given by

$$c = c_0[1 - (2\alpha\omega)^{-\frac{1}{2}} + \ldots]. \quad (3.14)$$

Rayleigh (1945, p. 319) attributes this result to Helmholtz.

On the other hand, if $\omega \gg \alpha c_0^2/\phi$ and ϕ is not equal to 1, then (3.9) and (3.10) simplify to

$$k = \left(\frac{\alpha\omega}{\phi}\right)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} \left\{1 + \frac{e^{\frac{1}{4}i\pi}}{(\alpha\omega)^{\frac{1}{2}}}\left(\frac{\phi}{\phi-1}\right)^{\frac{1}{2}} + \ldots\right\}, \quad (3.15)$$

$$k = \left(\frac{i\alpha\omega}{\phi} - \beta_{1n}^2\right)^{\frac{1}{2}} \{1 + [-i\alpha\omega(1 - \phi^{-1}) - \beta_{1n}^2]^{-\frac{1}{2}} + \ldots\}. \quad (3.16)$$

When $\phi = 1$ and $\omega \gg \alpha c_0^2/\phi$, $\kappa^2 \sim k^2 - i\alpha\omega$, so that the function $F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]$ is a common factor of (3.1). Then the root (3.15) disappears and the second term in (3.16) is absent.

We now consider the left side of (3.1) to be a small perturbation of the right-hand side. Then by proceeding as in the derivation of (3.10), we obtain for $\alpha\omega \gg 1$ and $\phi \neq 1$ the result

$$k = (i\alpha\omega - \beta_{1n}^2)^{\frac{1}{2}} \left[1 - \frac{\beta_{1n}^2\{i\alpha\omega[\alpha^2c_0^2 - i\alpha\omega(\phi-1)]/(\alpha^2c_0^2 - i\alpha\omega\phi) - \beta_{1n}^2\}^{\frac{1}{2}}}{(i\alpha\omega - \beta_{1n}^2)^2} + \ldots\right]. \quad (3.17)$$

If $\phi = 1$, the second term above vanishes and this root merges with the root (3.16), with no second term. When $\phi \neq 1$ and $\alpha\omega \gg \beta_{1n}^2$, (3.17) simplifies to

$$k = (\alpha\omega)^{\frac{1}{2}} e^{\frac{1}{4}i\pi} [1 + i\beta_{1n}^2/2\alpha\omega + \ldots]. \quad (3.18)$$

When the fluid is incompressible, we let $c_0 \to \infty$ in (3.1) to obtain

$$k^2F(k) = k^2F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]. \quad (3.19)$$

The roots of this equation can be obtained by letting $c_0 \to \infty$ in (3.4), which yields $k = 0$, in (3.8), which remains unchanged, in (3.9), which yields $k = 0$, in (2.10), which yields

$$k = i\beta_{1n}[1 + (-i\alpha\omega - \beta_{1n}^2)^{-\frac{1}{2}} + \ldots], \quad (3.20)$$

and in (3.17), which yields $k = (i\alpha\omega - \beta_{1n}^2)^{\frac{1}{2}}[1 - \beta_{1n}^2(i\alpha\omega - \beta_{1n}^2)^{-\frac{3}{2}} + \ldots]$, $n = 1, 2, \ldots$.

Numerical solutions of (3.1) have been obtained by Scarton & Rouleau (1973). They present curves of $\text{Re}\,k$ and $\text{Im}\,k$ as functions of ω/c_0 for the first 32 roots for three different values of αc_0 with $\phi = 1$. A superfluous root $k = (i\alpha\omega)^{\frac{1}{2}}$ was introduced in

Scarton & Rouleau (1973) because the extra factor $(k^2 - i\alpha\omega)^{\frac{1}{2}}$ was included in the dispersion equation. The asymptotic formulae in this section are in good agreement with their results where they are applicable. For example for $n = 1$ the exact numerical solution and the asymptotic result (3.20) yield the following values:

$$k = \begin{cases} -0{\cdot}0274864 + 3{\cdot}85879i & \text{[exact]} \\ -0{\cdot}027094 + 3{\cdot}85880i & \text{[asymptotic from (3.20)]} \end{cases} \quad \text{for} \quad \alpha\omega = 10^4,$$

$$k = \begin{cases} -0{\cdot}00860656 + 3{\cdot}84027i & \text{[exact]} \\ -0{\cdot}00856797 + 3{\cdot}84208i & \text{[asymptotic from (3.20)]} \end{cases} \quad \text{for} \quad \alpha\omega = 10^6.$$

The result (3.18) is also in good agreement with our numerical solution of (2.5) even for relatively small values of m. For example, for $m = 0{\cdot}1$, $\sigma = \frac{1}{2}$, $\omega = 2$, $Z_{ij} = 0$, $\Omega = 1$, $c_0 = \infty$ and $\alpha = 10^4$ we find

$$k = \begin{cases} 99{\cdot}558 + 100{\cdot}45i & \text{[exact]}, \\ 100(1+i) & \text{[asymptotic from (3.18)]}. \end{cases}$$

4. Viscous incompressible fluid in an unconstrained elastic tube

To obtain the dispersion equation for a viscous incompressible fluid in an unconstrained elastic tube, we set $Z_{ij} = 0$, $\Omega = 1$ and $c_0 = \infty$ in (2.5). Upon noting that $\kappa^2 \to k^2$ as $c_0 \to \infty$, we obtain

$$\begin{aligned} &\omega^4\{4m + 2m^2k^2(F(k) - F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]) + (2 + mk^2F(k))\,F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]\} \\ &\quad - \omega^2\{4mk^2 + 2k^2(F(k) - F[(k^2 - i\alpha\omega)^{\frac{1}{2}}])\,(m^2(k^2+1) - 4k^2\alpha^{-2} + i2m\omega\alpha^{-1}) \\ &\quad + k^2F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]\,(-4m\sigma + mF(k) - i2\omega\alpha^{-1}[4 - F(k)])\} \\ &\quad + 2k^2(F(k) - F[(k^2 - i\alpha\omega)^{\frac{1}{2}}])\,(m^2k^2(1-\sigma^2) + i2\omega\alpha^{-1}mk^2(1-2\sigma)) = 0. \end{aligned} \tag{4.1}$$

We shall first examine this equation when the viscosity is small, so that $\alpha \gg 1$.

Let us begin with very low frequencies, for which $\alpha\omega \ll 1$. By neglecting terms proportional to $\omega\alpha^{-1}$ and $k^2\alpha^{-2}$ in (4.1) and expanding $F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]$ for $\alpha\omega$ small, we get

$$\begin{aligned} &\omega^4\{4m + [2 + mk^2F(k)]F(k)\} - \omega^2\{4mk^2 + k^2F(k)\,(-4m\sigma + mF(k))\} \\ &\qquad + i\alpha\omega k^3F'(k)m^2(1-\sigma^2) \\ &= i\alpha\omega^3(2k)^{-1}F'(k)\,\{\omega^2[2 + mk^2(-2m + F(k))] + mk^2[2m(1+k^2) + 4\sigma - F(k)]\}. \end{aligned} \tag{4.2}$$

To find a root of (4.2) which is small, we use (3.3) for F, neglect the right side of (4.2) and neglect the ω^4 term on the left. Then we obtain the result

$$k_+ = 2\left(\frac{\omega}{\alpha}\right)^{\frac{1}{2}}\left[\frac{5-4\sigma}{m(1-\sigma^2)}\right]^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\left\{1 + i\alpha\omega\,\frac{(1-2\sigma)[3(3-2\sigma) - 2m(1-2\sigma)]}{(5-4\sigma)^2} + \dots\right\}. \tag{4.3}$$

This is a low frequency result, valid when ω is small compared with α^{-1}.

The leading term in (4.3) was given by Morgan & Kiely (1954), who recognized that it is valid only for $\omega \ll \alpha^{-1} \ll 1$. We shall show later that the leading term is valid also when $\omega \ll \alpha \ll 1$. It is the viscous analogue of the root denoted by k^+ in I. We note that, for small ω, k_+ as given by (4.3) behaves like $\omega^{\frac{1}{2}}$. The corresponding inviscid root behaves like ω, so it is not valid near $\omega = 0$. The accuracy of (4.3) is indicated by the following comparison of the first term in (4.3) with our numerical solution of (4.2) for $\omega = 10^{-6}$, $\alpha = 10^2$, $m = 0{\cdot}1$ and $\sigma = \frac{1}{2}$:

$$k = \begin{cases} (8{\cdot}9443 + 8{\cdot}94421i) \times 10^{-4} & \text{[exact]}, \\ 8{\cdot}944(1+i) \times 10^{-4} & \text{[asymptotic from (4.3)]}. \end{cases}$$

For the phase velocity c_+, the viscous root (4.3) yields

$$c_+ = \omega/\mathrm{Re}\, k = (\alpha\omega)^{\frac{1}{2}}\, [m(1-\sigma^2)/2(5-4\sigma)]^{\frac{1}{2}} + \ldots, \tag{4.4}$$

which vanishes at $\omega = 0$. The inviscid theory yields instead at $\omega = 0$ (I, equation (114); see also Atabek & Lew 1966)

$$c_{\pm}(0) = [2m(1-\sigma^2)]^{\frac{1}{2}} \{2 + m \pm [(2+m)^2 - 8m(1-\sigma^2)]^{\frac{1}{2}}\}^{-\frac{1}{2}}. \tag{4.4a}$$

When m is small, (4.4a) becomes Young's (1808) velocity

$$c_+(0) = c_Y = [\tfrac{1}{2}m(1-\sigma^2)]^{\frac{1}{2}}. \tag{4.4b}$$

In dimensional terms, (4.4b) is $c_Y' = (Eh/2\rho_0 a)^{\frac{1}{2}}$. Both (4.4$a$) and (4.4$b$) differ from the value zero given by (4.4) at $\omega = 0$ because viscosity is important near $\omega = 0$. This is clear in figure 1 (a), where c_+ is shown as a function of ω for various values of the viscosity parameter α.

There is another root k_- of (4.2) which is proportional to ω at low frequencies and which appears to have been overlooked in the literature. To find it we assume that $k = O(\omega)$ and retain those terms which are dominant for ω small. In this way we get the new result

$$k_- = \omega \left(\frac{4m+2}{m(5-4\sigma)}\right)^{\frac{1}{2}} \left(1 + \frac{i\alpha\omega}{4(2m+1)} \left[\frac{2-\sigma-m(1-2\sigma)}{5-4\sigma}\right]^2\right) + \ldots, \quad \omega \ll \alpha^{-1}. \tag{4.5}$$

This root is analogous to the root designated k^- in I. The phase velocity of the mode (4.5) is, at $\omega = 0$,

$$c_-(0) = \left(\frac{m(5-4\sigma)}{4m+2}\right)^{\frac{1}{2}}. \tag{4.6}$$

The velocity (4.6) differs from the inviscid zero-frequency result (4.4a). When m is small, that velocity reduces to Lamb's (1898) result $c_-(0) = c_L = 1$, or in dimensional form $c_L' = [E/\rho_1(1-\sigma^2)]^{\frac{1}{2}}$.

For $\omega = 10^{-6}$, $\alpha = 10^2$, $m = 0{\cdot}1$ and $\sigma = \frac{1}{2}$ the numerical solution of (4.2) and the first term of (4.5) compare as follows:

$$k = \begin{cases} 2{\cdot}8284 \times 10^{-6} - 1{\cdot}5014 \times 10^{-11} i & \text{[exact]}, \\ 2{\cdot}8284 \times 10^{-6} + 0i & \text{[asymptotic from (4.5)]}. \end{cases}$$

Next we consider low viscosities such that $\alpha^{-1} \ll 1$ and frequencies for which $\alpha\omega \gg 1$. Then it follows from (3.6) that for $|k|$ small $F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]$ is small. By neglecting terms of order α^{-1} and α^{-2} in (4.1), we can write this equation in the form

$$f(\omega, k) = m^{-1} g(\omega, k)\, F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]. \tag{4.7}$$

Here f and g are defined by

$$(\omega, k) \equiv \omega^4\{4 + 2mk^2F(k)\} - \omega^2\{4k^2 + 2m(k^2+1)\, k^2F(k)\} + 2m(1-\sigma^2)\, k^4F(k), \tag{4.8}$$

$$g(\omega, k) \equiv -\omega^4\{2 + mk^2F(k) - 2m^2k^2\} + \omega^2\{-2m^2k^2(k^2+1) + k^2m[-4\sigma + F(k)]\} + 2m^2(1-\sigma^2)\, k^4. \tag{4.9}$$

Since $f = 0$ is the dispersion equation in the inviscid case, (4.7) can be used to yield viscous corrections to the inviscid roots.

Let us begin with the low frequency case $1 \gg \omega \gg \alpha^{-1}$. From (4.7) we find the corrected roots for the two low frequency tube modes to be

$$k_{\pm} = \frac{\omega}{c_{\pm}(0)} + m^{-1} g[\omega, \omega/c_{\pm}(0)]\, F[\{\omega^2 c_{\pm}^{-2}(0) - i\alpha\omega\}^{\frac{1}{2}}] \Big/ \frac{\partial f}{\partial k}[\omega, \omega/c_{\pm}(0)] + \ldots, \quad (4.10)$$

with $c_{\pm}(0)$ given by (4.4*a*). By using (3.7) to simplify F, together with the definitions of and g, we can write (4.10) in the following more explicit form:

$$k_{\pm} = \frac{\omega}{c_{\pm}(0)} \left\{ 1 + A_{\pm} \frac{e^{\frac{1}{4}i\pi}}{(\alpha\omega)^{\frac{1}{2}}} + \frac{i}{\alpha\omega} (B_{\pm} - \tfrac{1}{2} A_{\pm}^2) + \ldots \right\}, \quad (4.11)$$

where

$$A_{\pm} = \frac{2m^2(1-\sigma^2) + m(1 - 4\sigma - 2m)\, c_{\pm}^2(0) - 2c_{\pm}^4(0)}{2m[2m(1-\sigma^2) - (2+m)\, c_{\pm}^2(0)]},$$

$$B_{\pm} = A_{\pm} \left\{ \frac{(2 - A_{\pm})\, 2m(1-\sigma^2) + (1 - 4\sigma - 2m)\, c_{\pm}^2(0)}{2m(1-\sigma^2) - (2+m)\, c_{\pm}^2(0)} - \tfrac{1}{2} \right\}.$$

When $m \ll 1$ in addition to the conditions $1 \gg \omega \gg \alpha^{-1}$, (4.11) simplifies to

$$\left.\begin{aligned} k_{+} &= \frac{\omega}{c_Y} \left\{ 1 + \frac{m\sigma^2}{4} + \left(1 - \frac{\sigma}{2}\right)^2 (\alpha\omega)^{-\frac{1}{2}} e^{\frac{1}{4}i\pi} - \frac{m\, e^{\frac{1}{4}i\pi}}{8} (\alpha\omega)^{-\frac{1}{2}} \right. \\ &\qquad \left. \times \left[4\sigma - 6\sigma^3 + \frac{5\sigma^4}{2} \right] + i \left(1 - \frac{\sigma}{2}\right)^2 (6 + 5\sigma)(2 - \sigma)(8\alpha\omega)^{-1} + \ldots \right\}, \\ k_{-} &= \frac{\omega}{c_L} \left\{ 1 - \frac{m\sigma^2}{4} + \left[\frac{1}{2m} + \sigma \left(1 - \frac{3\sigma}{8}\right) \right] (\alpha\omega)^{-\frac{1}{2}} e^{\frac{1}{4}i\pi} \right. \\ &\qquad \left. - \frac{i}{4m\alpha\omega} \left[\frac{1}{2m} + 1 - 2\sigma + \frac{3\sigma^2}{8} \right] + \ldots \right\}. \end{aligned}\right\} \quad (4.12)$$

The first and third terms in the result for k_+ were given by Morgan & Kiely (1954) and the special case $\sigma = 0$ was given by Witzig (1914). The result for k_- is new. From (4.12) we obtain

$$\left.\begin{aligned} c_{+} &= c_Y \left\{ 1 - \frac{m\sigma^2}{4} - \left(1 - \frac{\sigma}{2}\right)^2 (2\alpha\omega)^{-\frac{1}{2}} + \frac{m}{8(2\alpha\omega)^{\frac{1}{2}}} \left[4\sigma - 6\sigma^3 + \frac{5\sigma^4}{2} \right] + \ldots \right\}, \\ c_{-} &= c_L \left\{ 1 + \frac{m\sigma^2}{4} - \frac{1}{(2\alpha\omega)^{\frac{1}{2}}} \left[\frac{1}{2m} + \sigma \left(1 - \frac{3\sigma}{8}\right) \right] + \ldots \right\}. \end{aligned}\right\} \quad (4.13)$$

When the compressibility is finite, the terms c_0^{-2} and $\sigma^2 c_L/2c_0^2$ must be added to these expressions for c_+ and c_- respectively, as was shown in I, equations (115).

Womersley (1957) emphasized the importance of the dimensonless parameter $(\alpha\omega)^{\frac{1}{2}} = a(\rho_1 \omega'/\mu)^{\frac{1}{2}}$ in the study of pulse waves in mammalian circulatory systems. We see that this occurs in (4.13) and in other formulae, so we shall evaluate it for the human aorta. From McDonald (1974) we get the following typical values of the parameters: $a = 1{\cdot}1$ cm, $\rho_1 = 1{\cdot}06$ g cm^{-3}, $\sigma = 0{\cdot}5$, $\rho_0 = 1$ g cm^{-3}, $E = 5 \times 10^6$ dyn cm^{-2}, $h/a = 0{\cdot}1$ and $\mu = 0{\cdot}04$ P. We also choose $\omega' = 2\pi(\frac{70}{60})$ s^{-1}, which corresponds to the fundamental frequency of the heartbeat, which is about 70 cycles/min. Then we find that $m = 0{\cdot}1$, $\omega_0 = 2{\cdot}3 \times 10^3$ s^{-1}, $\omega = 3{\cdot}2 \times 10^{-3}$, $\alpha = 7{\cdot}0 \times 10^4$ and $\alpha\omega = 220$. Thus the conditions

$\omega \gg 1$, $\alpha \gg 1$, $\alpha\omega \gg 1$ and $m \ll 1$ are all well satisfied, so that (4.13) is valid. These conditions are also satisfied for the aortas of other large mammals (McDonald 1974).

These considerations enable us to explain the paradoxical result that Young's inviscid zero-frequency velocity formula (4.4*b*) fits well the observed pulse wave velocities in mammalian aortas, even though it differs significantly from the value $c = 0$ given by the viscous result (4.4) at $\omega = 0$. The explanation is that $\alpha\omega$ is large in mammalian aortas, so the zero-frequency result (4.3) is not valid, but (4.13) is instead. The leading term in c_+ is just c_Y, the viscous correction is just 2·7 % for the human aorta and the correction due to the term $\frac{1}{4}m\sigma^2$ is 0·6 %. The correction term due to compressibility is negligible as we see by setting $c_0' = 1570$ m/s, the value for horse serum (Goldman & Hueter 1956), since then $c_0^{-2} = 2\times 10^{-4}$. For higher harmonics ω and $\alpha\omega$ are even larger, so the result (4.13) is all the more accurate. However, for smaller arteries $\alpha\omega$ is smaller because the tube radius is smaller. Then the role of viscosity is more important and ultimately Young's formula becomes invalid.

Next, we consider high frequencies, $\omega \gg 1$, as well as low viscosity, $\alpha \gg 1$. Then if $k = O(\omega)$, (4.7) becomes

$$\omega^2[4m + 2m^2k^2\{F(k) - F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]\}](\omega^2 - k^2) = -m\omega^4k^2F(k)\,F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]. \tag{4.14}$$

The right side of (4.14) is small compared with the left side. If the right side is omitted, one root of the resulting equation is $\omega = k$. Because of the right side, this root becomes

$$k_- = \omega\left[1 + \frac{1}{2m\omega[(1 - i\alpha/\omega)^{\frac{1}{2}} - 1]} + \ldots\right]. \tag{4.15}$$

We have denoted it by k_- because it is the continuation of the root we previously called k_-. There is another root of (4.14), which is the continuation of k_+. It is given approximately by (3.18) with $\beta_{1n} = 0$. If $\alpha \gg \omega \gg 1$, (4.15) becomes

$$k_- = \omega[1 + e^{\frac{1}{4}i\pi}/2m(\alpha\omega)^{\frac{1}{2}} + \ldots]. \tag{4.16}$$

On the other hand, if $\omega \gg \alpha \gg 1$, (4.15) becomes instead

$$k_- = \omega[1 + i/m\alpha + \ldots]. \tag{4.17}$$

The phase velocity for this mode is $c_- = c_L + \ldots = 1 + \ldots$, which is also its value at intermediate low frequencies.

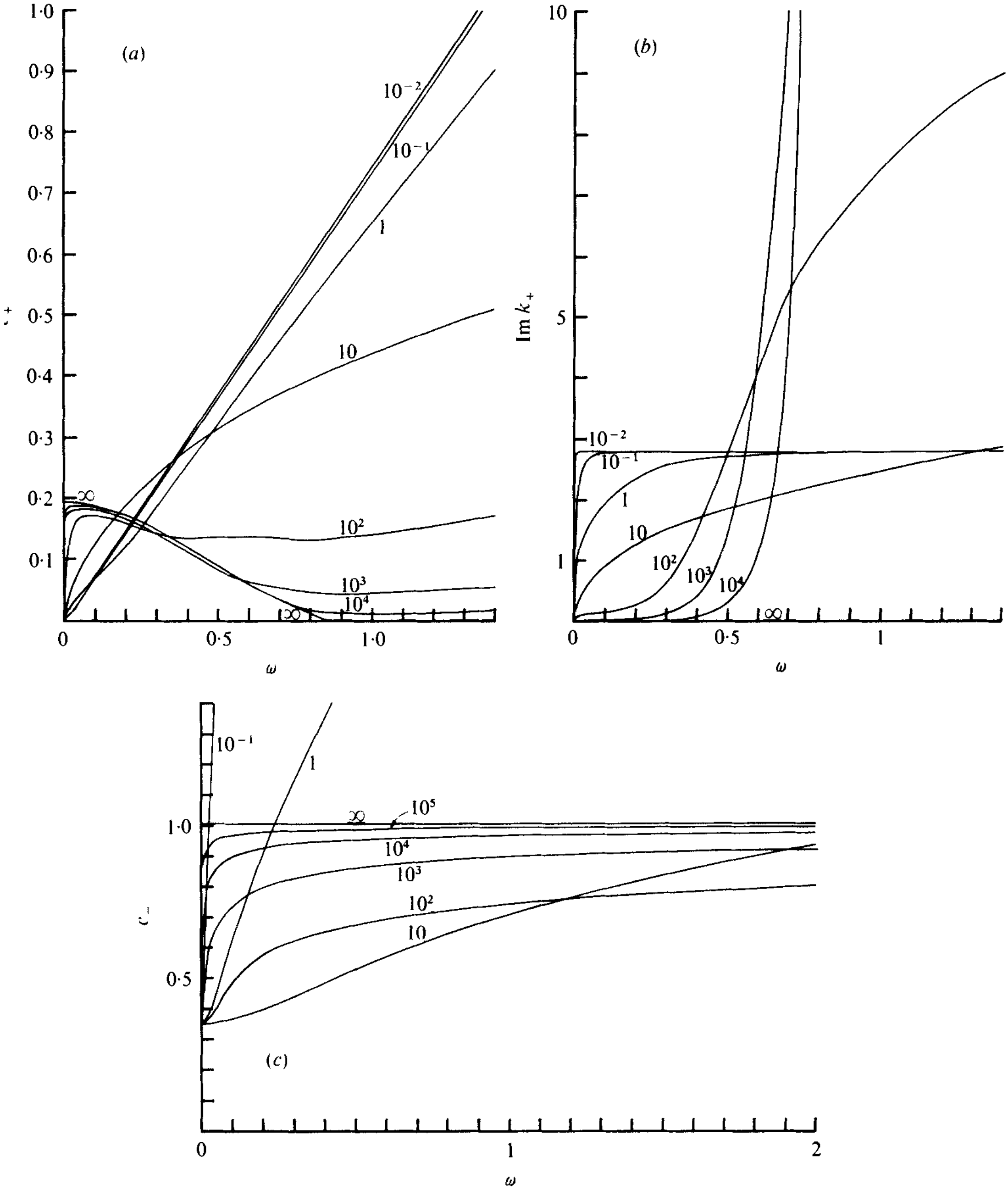
The accuracy of (4.16) is indicated by the following comparisons with our numerical solutions of (4.1) for $\omega = 2$, $\alpha = 10^4$, $\sigma = \frac{1}{2}$ and two values of m:

$$k = \begin{cases} 2\cdot0462 + 0\cdot052892i & \text{[exact]} \\ 2\cdot05 + 0\cdot05i & \text{[asymptotic from (4.16)]} \end{cases} \quad \text{for} \quad m = 0\cdot1,$$

$$k = \begin{cases} 1\cdot9293 + 9\cdot838\times10^{-4}i & \text{[exact]} \\ 2\cdot0 + 5\cdot0\times10^{-4}i & \text{[asymptotic from (4.16)]} \end{cases} \quad \text{for} \quad m = 10.$$

Let us now consider a very viscous fluid, $\alpha \ll 1$, at low frequencies, $\omega \ll 1$. If $\omega \ll \alpha \ll 1$ and $k = O[(\omega/\alpha)^{\frac{1}{2}}]$, then (4.2) becomes

$$\omega^2k^2[m(5 - 4\sigma) - 6i\omega/\alpha] + \tfrac{1}{4}k^4m[i\alpha\omega m(1 - \sigma^2) - 2\omega^2(1 - 2\sigma)] + \ldots = 0$$



FIGURES 1 (*a–c*). For legend see next page.

and its root is given by

$$k = 2\left(\frac{\omega}{\alpha}\right)^{\frac{1}{2}}\left[\frac{(5-4\sigma)}{m(1-\sigma^2)}\right]^{\frac{1}{2}} e^{\frac{1}{4}i\pi}\left\{1-\frac{i\omega}{\alpha m}\frac{(4-5\sigma)(2-\sigma)}{(5-4\sigma)(1-\sigma^2)}+\dots\right\}. \tag{4.18}$$

We note that the leading term above is the same as the leading term in (4.3), which justifies the remark following that equation.

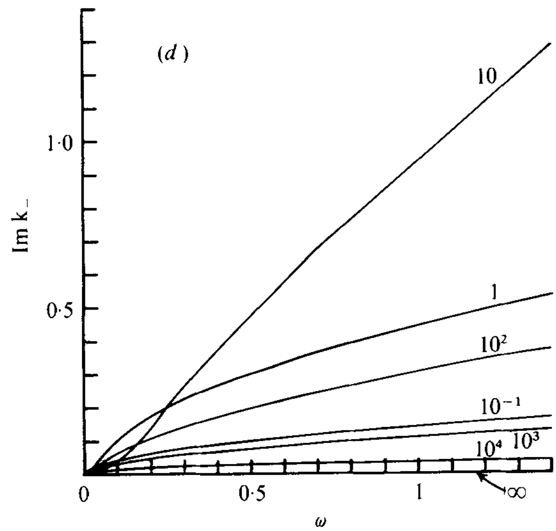


FIGURE 1. (*a*), (*c*) The non-dimensional phase velocity $c_\pm$ and (*b*), (*d*) the imaginary part of the propagation constant Im $k_\pm$ are shown as functions of the non-dimensional frequency ω for waves in an unconstrained thin elastic tube filled with a viscous incompressible fluid. The curves are based on numerical solution of (4.1) for various values of the non-dimensional viscosity parameter α ranging from 10^{-2} to ∞. Other parameter values are $\sigma = 0{\cdot}5$ and $m = 0{\cdot}1$, which are representative of mammalian blood in arteries. The curves labelled ∞ represent the inviscid result. For α finite, c_+ and Im $k_\pm$ approach the origin in accordance with (4.4), (4.3) and (4.5) while $c_-(0)$ is given by (4.6). For the inviscid curve, $c_\pm(0)$ are given by (4.4*a*) or (4.4*b*). For the curves labelled 10^2, 10^3 and 10^4 (slightly viscous fluid), the values of $c_\pm$ and Im $k_\pm$ following the initial rise from the origin are represented by (4.13) and (4.12), respectively. The behaviour of these curves for $\omega > 1$ is represented by (3.18) with $\beta_{1n} = 0$. The curves labelled 10^{-1} and 10^{-2} (very viscous fluid) approach the origin in accordance with (4.19) and (4.20). For large frequencies, the curves for c_+ increase linearly with ω and the curves for Im k_+ are constant, both in accordance with (4.24) for $n = 1$.

If $\omega \ll \alpha \ll 1$ and $k = O(\omega)$, (4.2) becomes

$$(4m+2)\,\omega^4 - \omega^2 k^2[m(5-4\sigma) - i6\omega/\alpha] + \ldots = 0,$$

for which a root is

$$k_- = \omega\left[\frac{4m+2}{m(5-4\sigma)}\right]^{\frac{1}{2}}\left[1 + \frac{i3\omega}{m(5-4\sigma)\,\alpha} + \ldots\right]. \tag{4.19}$$

If instead $\alpha \ll \omega \ll 1$ and $k = O[(\alpha\omega)^{\frac{1}{2}}]$, (4.2) yields

$$(4m+2)\,\omega^4 + i6\omega^3\alpha^{-1}k^2 + \ldots = 0.$$

This has the root

$$k_- = [\tfrac{1}{3}(2m+1)]^{\frac{1}{2}}\,(\alpha\omega)^{\frac{1}{2}}\,e^{\frac{1}{4}i\pi} + \ldots. \tag{4.20}$$

The corresponding phase velocity is

$$c_- = [6\omega/(2m+1)\,\alpha]^{\frac{1}{2}} + \ldots. \tag{4.21}$$

When $\alpha \ll 1$, $\omega \ll 1$ and $|k| \gg 1$, the dispersion equation (4.1) simplifies to

$$2k^2\{F(k) - F[(k^2 - i\alpha\omega)^{\frac{1}{2}}]\}\left\{-\frac{4k^2}{\alpha^2} + im\frac{2\omega}{\alpha}\right\} - \frac{i2\omega}{\alpha}k^2F[(k^2 - i\alpha\omega)]\,[4 - F(k)] = 0. \tag{4.22}$$

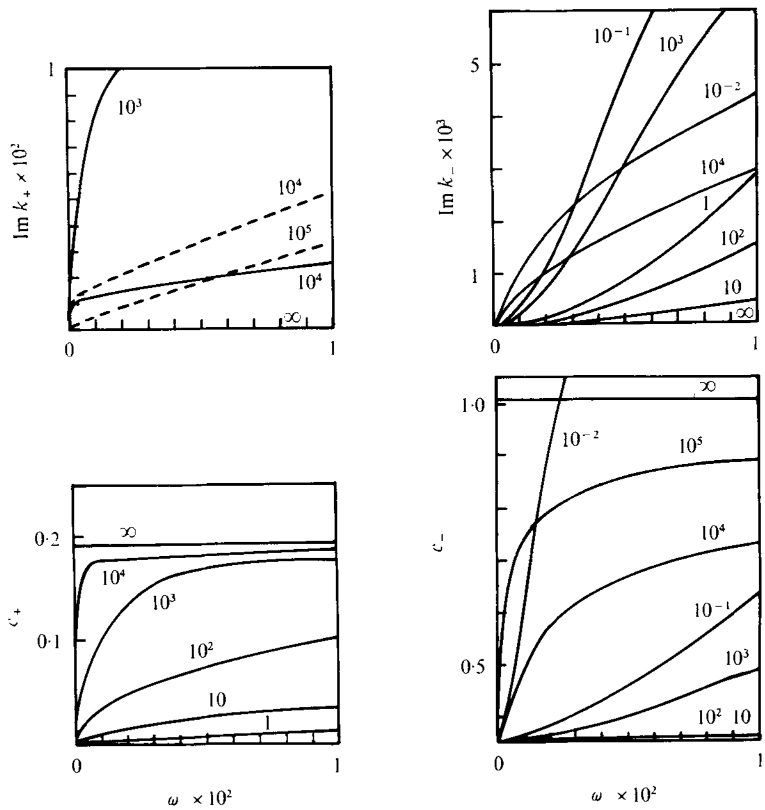


FIGURE 2. The same quantities representing waves in an unconstrained thin elastic tube filled with a viscous incompressible fluid plotted in figure 1 are shown on an enlarged scale that encompasses the frequency range of principal physiological interest. The two dashed lines show, for comparison purposes, the effect of viscoelasticity when $\gamma = 0{\cdot}1$.

Because $\alpha\omega \ll 1$, (4.22) can be reduced to $k^2[4-(k^2+1)F^2(k)]+\ldots = 0$. Then (3.6) can be used to yield

$$4k+e^{2k-\frac{1}{2}i\pi}+\ldots = 0. \tag{4.23}$$

The solutions of (4.23) are

$$k_n = \tfrac{1}{2}\log(4n\pi)+in\pi+\ldots, \quad n = 1,2,\ldots. \tag{4.24}$$

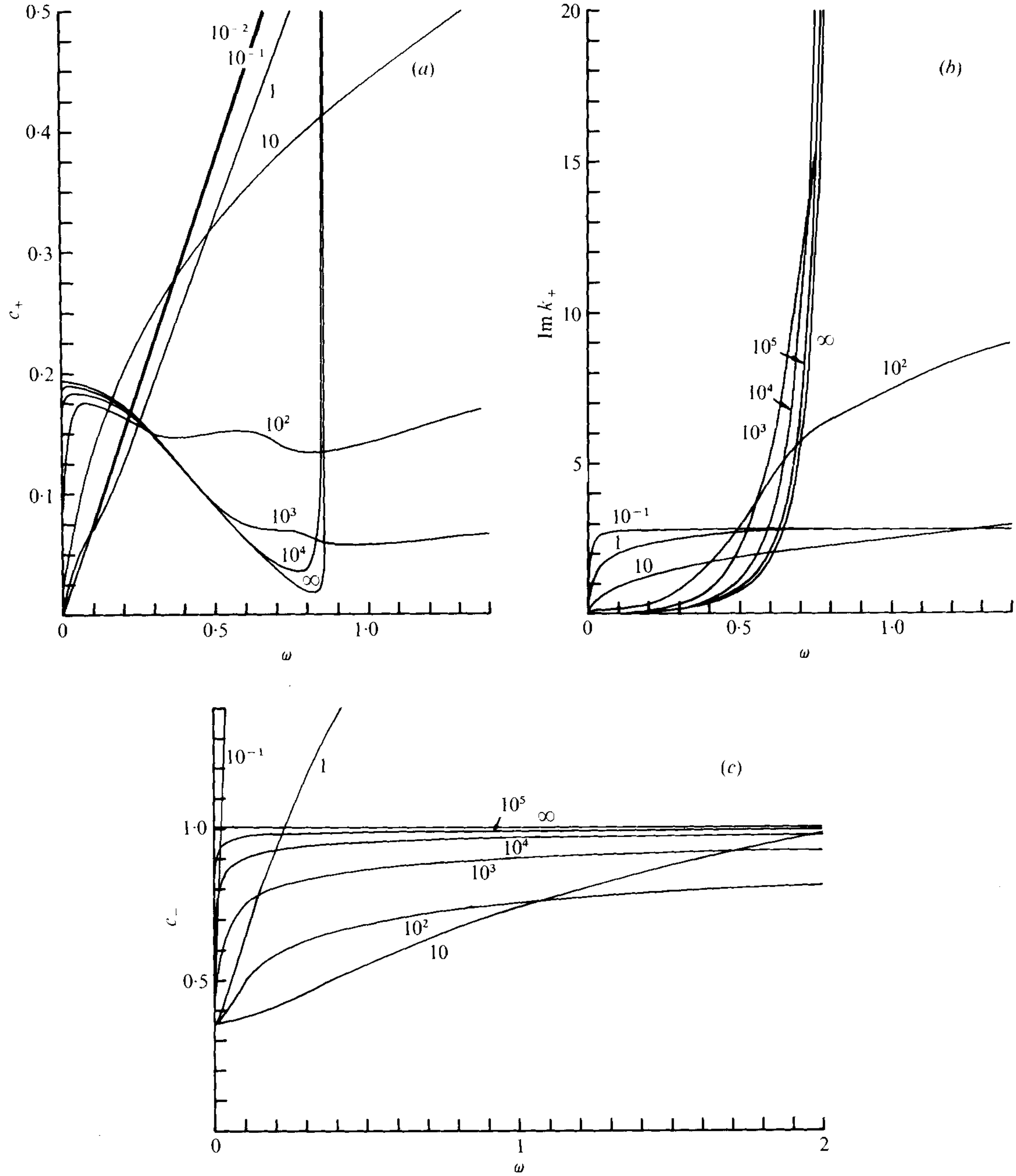
At high frequencies for which $\omega \gg \alpha^{-1}$, (4.1) becomes

$$k^2F(k)\,[m^2\omega^4+4\omega^2k^2\alpha^{-2}]+\ldots = 0. \tag{4.25}$$

The roots of the factor $k^2F(k)$ in (4.25) are $k = 0$ and $k = i\beta_{1n}$, which are given by (3.12) and (3.13) with $c_0 = \infty$ and $\alpha\omega \gg 1$. In addition there is the root

$$k = \tfrac{1}{2}im\alpha\omega+\ldots. \tag{4.26}$$

In figure 1 we show the quantities $c_\pm$ and Im $k_\pm$ as functions of the frequency ω for various values of the viscosity parameter α. These curves have been obtained by numerical solution of the dispersion equation (4.1). The exact numerical solutions agree very well with the asymptotic results derived herein, in the frequency domains in which they are applicable. In figure 2, the same quantities are shown on an enlarged scale for the frequency range of principal physiological interest: $0 \leqslant \omega \leqslant 10^{-2}$.



FIGURES 3 (*a*–*c*). For legend see next page.

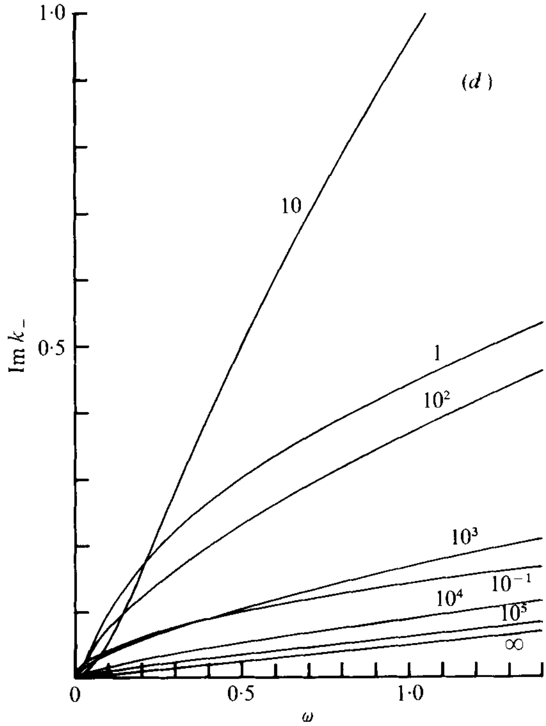


FIGURE 3. (*a*) c_+, (*b*) Im k_+, (*c*) c_- and (*d*) Im k_- *vs.* ω for waves in an unconstrained thin viscoelastic tube filled with a viscous fluid. The parameter values are $m = 0{\cdot}1$, $\sigma = 0{\cdot}5$ and $\gamma = 0{\cdot}1$. The labels on the curves denote the values of α.

5. Viscoelastic effects

When the tube is viscoelastic, the viscoelastic parameter γ is not zero and therefore $\Omega = e^{-i\gamma}$ is not unity. If we write the dispersion equation (2.5) in the form $D = 0$, we see that D satisfies the following identity, in which $\Omega^* = e^{i\gamma}$:

$$D[k, \omega^2, c_0^2, \alpha^{-2}, Z_{ij}, \Omega] = \Omega^2 D[k, \Omega^*\omega^2, \Omega^* c_0^2, \Omega^*\alpha^{-2}, \Omega^* Z_{ij}, 1]. \quad (5.1)$$

On the right side we have the dispersion function for the elastic case with $\omega^2, c_0^2, \alpha^{-2}$ and Z_{ij} each multiplied by Ω^*. Therefore the solution of $D = 0$ in the viscoelastic case is just the solution k for the elastic case with these replacements. We note that $\alpha\omega$, $\alpha^2 c_0^2$, ω^2/c_0^2 and κ^2 are unchanged in this process. Therefore all the results of §4 hold when the appropriate arguments in them have been multiplied by Ω^*.

As an example of particular significance for mammalian circulatory systems, let us consider the case of an incompressible fluid with $\alpha^{-1} \ll \omega \ll 1$ and $m \ll 1$. This corresponds to low frequency waves in a fluid of low viscosity in a light tube. With $Z_{ij} = 0$ the tube is unconstrained. Then (4.12) applies with ω multiplied by $\Omega^{*\frac{1}{2}} = e^{\frac{1}{2}i\gamma}$ and $\alpha\omega$ unchanged. The corresponding phase velocities are

$$\left.\begin{aligned} c_+ &= c_Y \sec\tfrac{1}{2}\gamma[1 - \tfrac{1}{4}m\sigma^2 - (1 - \tfrac{1}{2}\sigma)^2 (2\alpha\omega)^{-\frac{1}{2}} (1 - \tan\tfrac{1}{2}\gamma) + \ldots], \\ c_- &= c_L \sec\tfrac{1}{2}\gamma[1 + \tfrac{1}{4}m\sigma^2 - \{1/2m + \sigma(\sigma - \tfrac{3}{8}\sigma)\} (2\alpha\omega)^{-\frac{1}{2}} (1 - \tan\tfrac{1}{2}\gamma) + \ldots]. \end{aligned}\right\} \quad (5.2)$$

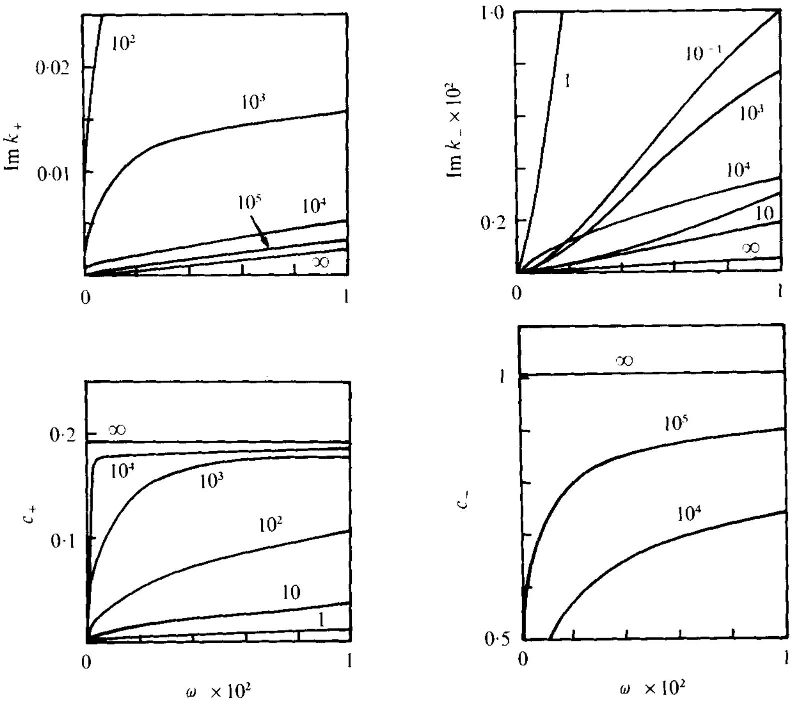


FIGURE 4. The same as figure 2 but for a viscoelastic tube with $\gamma = 0{\cdot}1$.

In figures 3(a)–(d) graphs of c_+, $\operatorname{Im} k_+$, c_- and $\operatorname{Im} k_-$ as functions of ω based upon direct numerical solution of the dispersion equation (2.5) are shown. Figure 4 displays the same quantities in the frequency range of physiological interest for $\gamma = 0{\cdot}1$.

6. Impedance, compliance and wall impedance

It is often convenient to consider the volume flux $Q' = a^3\omega_0 Q$ of fluid through a cross-section of the tube and the average pressure $P' = \rho_0 a^2 \omega_0^2 P$ over a cross-section. For the solution (2.4) the dimensionless flux Q and the average pressure P are

$$Q(z,t) = 2\pi \int_0^1 w(r)\, r\, dr = 2\pi \left[w_0 (k^2 - i\alpha\omega)^{-\frac{1}{2}} I_1[(k^2 - i\alpha\omega)^{\frac{1}{2}}] - \frac{p_0 \omega k}{c_0^2(\kappa^2 - k^2)\kappa} I_1(\kappa) \right] e^{i(kz-\omega t)}, \quad (6.1)$$

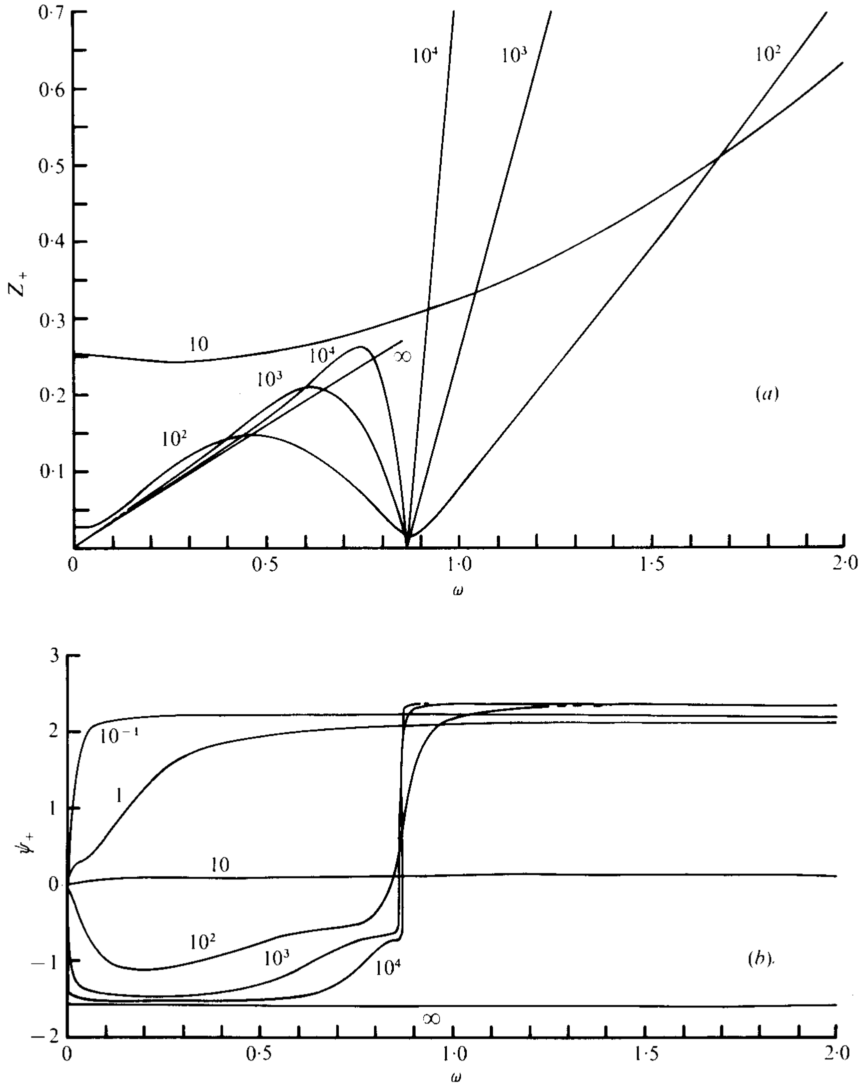
$$P(z,t) = 2\int_0^1 p(r)\, r\, dr = 2p_0 \kappa^{-1} I_1(\kappa)\, e^{i(kz-\omega t)}. \quad (6.2)$$

Because of their exponential dependence upon z and t, these functions satisfy a pair of differential equations called transmission-line equations, which can be written in the following forms:

$$P_z = -LQ_t - RQ = -(L - R/i\omega)\, Q_t = -(R - i\omega L)\, Q, \quad (6.3)$$

$$Q_z = -\tilde{C} P_t - GP = -(\tilde{C} - G/i\omega)\, P_t = -(G - i\omega\tilde{C})\, P. \quad (6.4)$$

The constants R, L, G and $\tilde{C}$ are respectively the hydraulic resistance, the fluid inertia, the seepage conductance and the compliance. Since all these quantities may depend



FIGURES 5 (*a*, *b*). For legend see next page.

upon ω, it is convenient to combine them into the impedance Z and the complex compliance C as follows:

$$Z = R - i\omega L = |Z|\, e^{i\Psi}, \tag{6.5}$$

$$C = \tilde{C} - G/i\omega = |C|\, e^{i\chi}. \tag{6.6}$$

Sometimes C is written in terms of the wall impedance Z_w and the sound speed in the fluid in the form $C = \pi c_0^{-2} - \pi/i\omega Z_w$. The dimensional impedances are

$$Z' = \rho_0\, \omega_0\, a^{-2} Z \quad \text{and} \quad Z'_w = \rho_0\, \omega_0\, Z_w,$$

while the dimensional compliance is $C' = C/\rho_0\, \omega$.

By using (6.1) and (6.2) in (6.3) and (6.4), we can obtain expressions for Z and Z_w in terms of the ratio p_0/w_0. This ratio is determined by the homogeneous linear equations

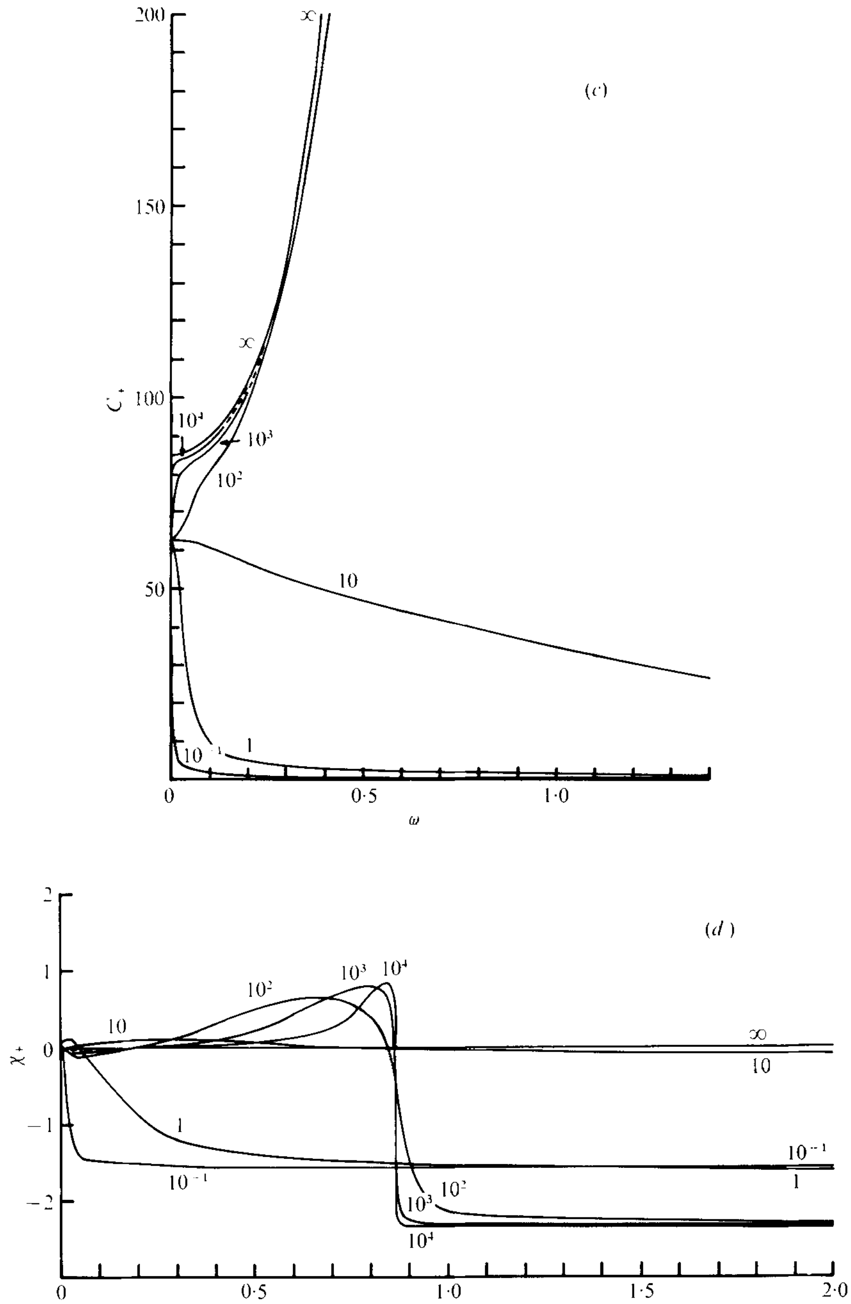


FIGURE 5. The quantities (a) $|Z_+|$, (b) Ψ_+, (c) $|C_+|$ and (d) χ_+, defined by (6.5)–(6.8) and associated with the root k_+, are shown as functions of the frequency for various values of α for the case of an incompressible fluid in an unconstrained elastic tube ($\gamma = 0$). Here $m = 0{\cdot}1$, $\sigma = 0{\cdot}5$.

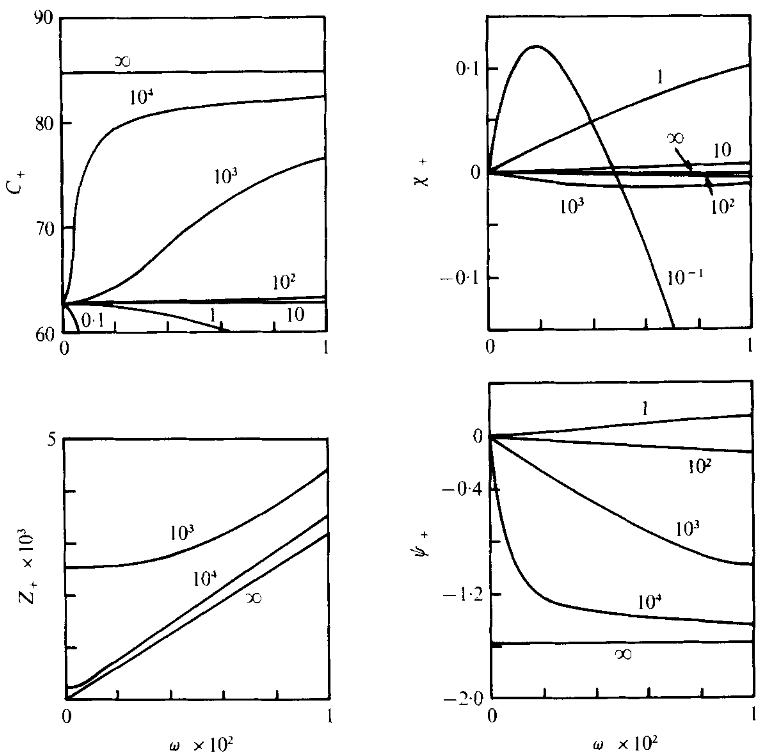


FIGURE 6. The same quantities as in figure 5 are shown on an enlarged scale in the frequency range of principal physiological interest.

for p_0, w_0, ξ_0 and ζ_0 given in I, which follow from the boundary conditions. When these equations are used to get this ratio and it is used in the expression (6.5) for Z, the result is

$$\begin{aligned} Z = {} & ic_0^2(\kappa^2-k^2)F(\kappa)(\pi\omega)^{-1}\{2m(\omega^2-k^2\Omega-Z_{22})+F[(k^2-i\alpha\omega)^{\frac{1}{2}}][\omega^2+2i\omega k^2\alpha^{-1} \\ & +mk(k\sigma\Omega+iZ_{21})]\} \\ & \times\{2m(\omega^2-k^2\Omega-Z_{22})(F(\kappa)-F[(k^2-i\alpha\omega)^{\frac{1}{2}}])+F(\kappa)F[(k^2-i\alpha\omega)^{\frac{1}{2}}] \\ & \times[\omega^2+2i\omega\alpha^{-1}(k^2-\kappa^2)+m(k^2-\kappa^2)(\sigma\Omega-iZ_{21}k^{-1})]\}^{-1}. \end{aligned} \tag{6.7}$$

Rather than writing out the expression for Z_w, we can express it in terms of Z by eliminating Q from (6.3) and (6.4) and then using (6.2) for P. This yields

$$\frac{k^2}{\omega^2} = \frac{Z}{-i\omega}\left[\frac{\pi}{-i\omega Z_w}+\frac{\pi}{c_0^2}\right] = \frac{ZC}{-i\omega}. \tag{6.8}$$

We can obtain Z_w or C from (6.8). Alternatively, if Z_w is computed directly from (6.4) in the same way as Z was, then (6.8) can be recognized as the dispersion equation (2.5).

The first transmission-line equation (6.3) is just the equation for the z component of momentum of the fluid, while (6.4) is the equation of mass conservation for the fluid. To derive (6.4) we denote the dimensionless cross-sectional area of the tube by

$$A = \pi(1+\xi)^2,$$

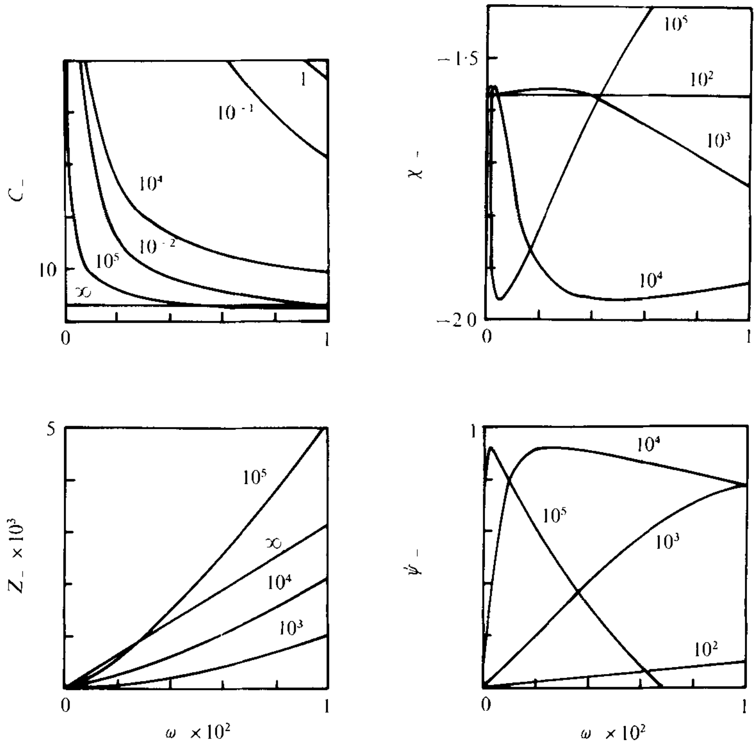


FIGURE 7. The quantities $|Z_-|$, Ψ_-, $|C_-|$ and χ_+ associated with the root k_-, as functions of ω for various values of α. Other quantities as in figure 5.

with the dimensional area $A' = Aa^2$. Then for ξ small, $A_t = 2\pi\xi_t$. From (2.3), (2.4) and the boundary conditions, ξ can be expressed in terms of P. Then we can write the last relation in the form

$$A_t = \frac{\pi}{-i\omega Z_w} P_t. \tag{6.9}$$

If we derive (6.9) in this way, we get an explicit expression for Z_w. Next we use the continuity equation for the fluid, which is $Q_z + \pi c_0^{-2} P_t + A_t = 0$. When (6.9) is used in this equation, it becomes (6.4).

The transmission-line equations (6.3) and (6.4) have been given by many other authors considering pulsatile blood flow through an artery, with simpler formulae for Z and Z_w (reviewed in Cox 1969). We shall see that some of them follow from (6.7) and (6.8) in limiting cases. Before doing so we note that Z and Z_w depend upon ω and k, and that k must be a root of the dispersion equation. Therefore the values of Z and Z_w depend upon the mode of propagation corresponding to the root k. We also note that the quantities Q and P are both zero for the non-axially symmetric modes, which are proportional to $e^{in\theta}$ with $n \neq 0$. Therefore the transmission-line equations for these modes must be formulated in terms of other quantities proportional to w and p.

Let us examine the impedance Z for an incompressible fluid in a rigid tube. To do so we let $c_0 \to \infty$ and $m \to \infty$ in (6.7) with $\kappa \to k$ and $c_0^2(\kappa^2 - k^2) \to -\omega^2$. One of the roots of the dispersion equation is $k = 0$, and for this root (6.7) yields

$$Z = -i\omega/\pi\{1 - F[(-i\alpha\omega)^{\frac{1}{2}}]\}, \quad c_0 = \infty, \quad m = \infty, \quad k = 0. \tag{6.10}$$

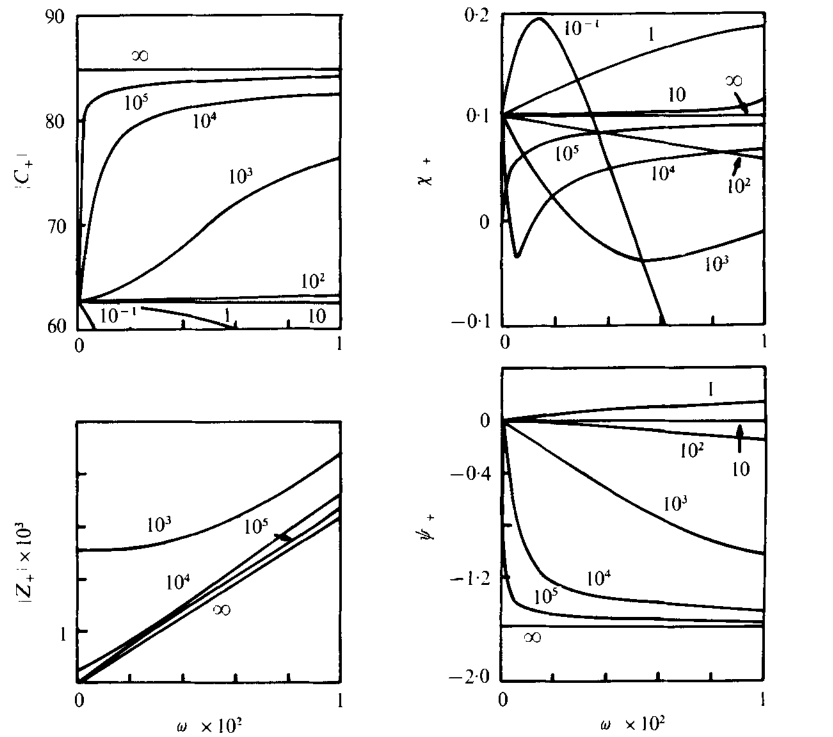


FIGURE 8. The same quantities as in figure 6 for the viscoelastic case $\gamma = 0{\cdot}1$.

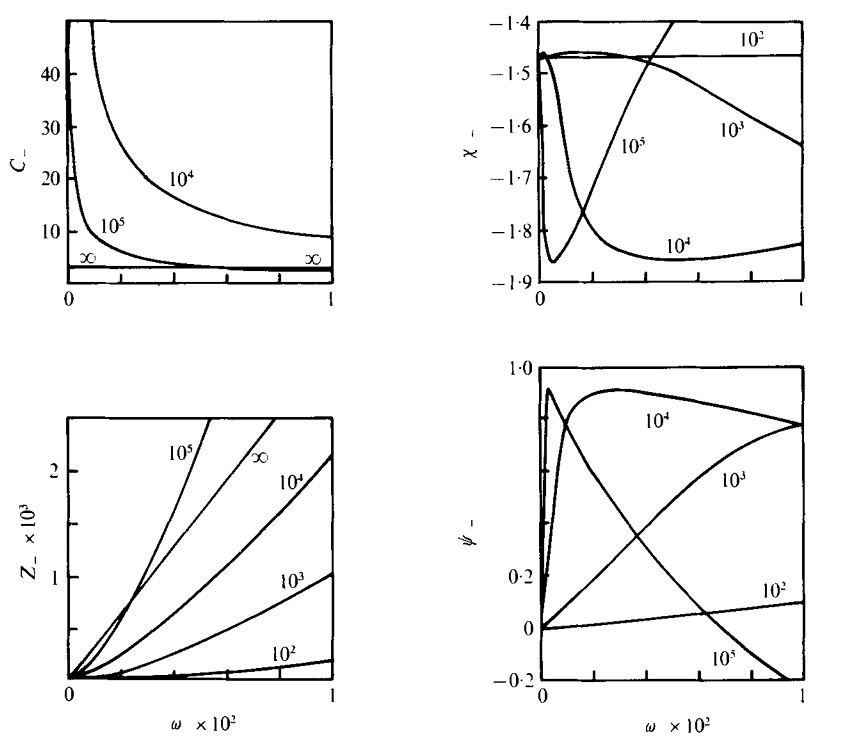


FIGURE 9. The same quantities as in figure 7 for the viscoelastic case $\gamma = 0{\cdot}1$.

Now since Z is finite and $k = 0$, (6.8) yields

$$C = 0, \quad Z_w = \infty. \tag{6.11}$$

The result (6.11) was to be expected in view of (6.9), since $A_t = 0$ for a rigid tube. If $\alpha\omega \gg 1$, which is the case for an inviscid fluid or for a viscous fluid at high frequencies, (6.10) simplifies to

$$Z = -\omega/\pi. \tag{6.12}$$

Next we shall consider an incompressible fluid in an unconstrained viscoelastic tube. In this case also $c_0 \to \infty$, $\kappa \to k$, $c_0^2(\kappa^2-k^2) \to \infty$ and in addition $Z_{ij} = 0$. Then (6.7) becomes

$$Z = -\frac{i\omega F(k)}{\pi}\left\{\frac{2m(\omega^2-k^2\Omega)+F[(k^2-i\alpha\omega)^{\frac{1}{2}}][\omega^2+2i\omega k^2\alpha^{-1}+m\sigma k^2\Omega]}{2m(\omega^2-k^2\Omega)\{F(k)-F[(k^2-i\alpha\omega)^{\frac{1}{2}}]\}+F(k)F[(k^2-i\alpha\omega)^{\frac{1}{2}}]\omega^2}\right\}. \tag{6.13}$$

If $\alpha^{-1} \ll \omega \ll 1$ and $m \ll 1$, two roots of the dispersion equation are $k_\pm$ as given by (4.12) with ω multiplied by $\Omega^{*\frac{1}{2}} = e^{\frac{1}{2}\gamma}$ and with $\alpha\omega$ unchanged. Then $F(k) = 1+O(\omega^2)$ and $F[(k^2-i\alpha\omega)^{\frac{1}{2}}] = 2(\alpha\omega)^{-\frac{1}{2}}e^{\frac{1}{4}i\pi} - i(\alpha\omega)^{-1} + O[(\alpha\omega)^{-\frac{3}{2}}]$. We denote the corresponding values of Z by $Z_\pm$ and simplify (6.13) to the form

$$Z_+ = -\frac{i\omega}{\pi}\left\{1+\frac{2e^{\frac{1}{4}i\pi}}{(\alpha\omega)^{\frac{1}{2}}}\left[1+\frac{\sigma}{2(c_Y^2-1)}\right]+\ldots\right\}, \tag{6.14}$$

$$Z_- = \frac{i\omega}{\pi}\left\{\frac{\begin{array}{l}2m[c_-^2(0)-1]+2(\alpha\omega)^{-\frac{1}{2}}e^{\frac{1}{4}i\pi}[c_-^2(0)+m\sigma-2mA_-]\\ \qquad +i(\alpha\omega)^{-1}[m\sigma(4A_--1)-4mB_--c_-^2(0)]\end{array}}{\begin{array}{l}2m[c_-^2(0)-1]+2(\alpha\omega)^{-\frac{1}{2}}e^{\frac{1}{4}i\pi}[c_-^2(0)-2m(A_-+c_-^2(0)-1)]\\ \qquad +i(\alpha\omega)^{-1}[-2m(2B_--4A_-+1-c_-^2(0))-c_-^2(0)]\end{array}}+\ldots\right\}. \tag{6.15}$$

From (6.8), since $c_0 = \infty$ we have $Z_w = -\pi Z k^{-2}$ and $C = k^2/i\omega Z$. Upon using (4.12), (6.14) and (6.15) in these relations, we get

$$C_+ = \frac{-\pi}{i\omega Z_{w+}} = \frac{\pi\Omega^*}{c_Y^2}\left\{1+\frac{m\sigma^2}{2}+\frac{2e^{\frac{1}{4}i\pi}}{(\alpha\omega)^{\frac{1}{2}}}\left[\frac{\sigma^2}{4}-\sigma+\frac{\sigma}{2(1-c_Y^2)}\right]+\ldots\right\}, \tag{6.16}$$

$$C_- = i\omega\Omega^*/c_-^2(0)\,Z_-. \tag{6.17}$$

The results (6.11) and (6.16) (with $\gamma = 0$ and only the leading term retained) have been used by Womersley (1957) and Skalak (1972) in their models of arterial blood flow.

The quantities $|Z_+|$, Ψ_+, $|C_+|$ and χ_+, which are defined by (6.5)–(6.8) and associated with the mode of propagation corresponding to the root k_+, are shown as functions of the frequency in figures 5(*a*)–(*d*) for the case of a viscous incompressible fluid in an unconstrained elastic tube. The same quantities and those associated with the root k_- are shown as functions of the frequency in the physiological range for the elastic case in figures 6 and 7 and for the viscoelastic case in figures 8 and 9.

This work was supported in part by the National Science Foundation under grant MPS 74-19234 and grant MCS 75-09837.

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